

Application of Haar wavelet-packets to the solution of linear and nonlinear Integral equations

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Abstract: The paper describes an innovative alternative to Walsh series and single term methods for the solution of linear and nonlinear integral equations using Haar wavelet packets. The underlying theory and properties of Haar wavelet packets are presented in some detail. The integral operators are expressed in wavelet packet bases resulting invariably into sparse matrices. The proposed methods, as a consequence, reduce the computation time and effort, concurrently achieving better accuracy in solving different kinds of integral equations. The error estimates are derived to establish the rapidity of convergence. A good agreement between computed results and the exact solutions is demonstrated using numerical tables and a computer generated graph. The methods are in general, conceptually simple, easy to implement, yield accurate results and more importantly, they have universal features for their applications to a wide range of problems, with higher order convergence rate.

Keywords: Block-pulse solution, Haar wavelet packets, Multi-Resolution Analysis, operation matrix, Series and single term methods, Walsh functions, Wavelet decomposition.

Introduction

Integral equations are well-known mathematical tools in the formulation of physical and mechanical problems. They arise in many branches of science, for example, in potential theory, acoustics, elasticity, fluid mechanics, theory of population etc. Any integral equation to be numerically solved can be reduced to a finite dimensional problem or discretized. There are two approaches to the discretization of linear integral equations. In one, often called the Galerkin (projection) method, expansions of functions and kernels involved are truncated in some basis and the resulting system of algebraic equations is solved numerically. In the second, developed by Nystrom, the integral operator is approximated yielding again a system of equations to be solved numerically [1]. Projection methods have been around for a long time. They have been modified in several ways. A more important modification is the *Sloan iteration*. Projection methods have also been used to solve nonlinear integral equations. Nonlinear integral equations have also been solved with generalization of the Nystrom method. But, these methods of historical interest impose severe restrictions on the underlying functions and integral operators and differ drastically in their requirements [2]. Projection methods for example, require mean convergence of the relevant expansions and hence impose conditions of the classical L^2 -theory, whereas Nystrom method works on the assumption that the functions and kernels are continuous (Beylkin (1992)). This ultimately limits their applicability to a large class of problems. Moreover, these methods converge too slowly due to their low approximation order, to be of much use in solving integral equations. This motivates search for suitable alternative is to solve integral equations.

There has been considerable revival of interest in solving differential and integral equations using techniques which involve Walsh functions for quite some time (Hsiao and Chen (1979)). More recently, in the field of numerical analysis, Walsh series methods such as, truncated series and single term have been successfully used for the numerical solution of several classes of problems, particularly linear and nonlinear integral equations (Sloss and Blyth(2003), Sepehrian and Razzaghi (2005)).

The use of wavelet methods in numerical analysis has opened up floodgates to several areas of research in recent years. In general, wavelet methods have been successfully used in the numerical treatment of differential and integral equations for three main tasks: One, preconditioning large systems arising from discretization of elliptic pde's [17], two, adaptive approximations of functions (operators) and finally sparse representation of initially dense matrices arising from the discretization of integral equations. The sparse representation of initially dense matrices arising in the discretization of integral equations via wavelet bases, leads to new methods for solution of integral equations. More precisely, integral operators when expressed in wavelet basis result in sparse matrices containing only $O(n \log n)$ non-negligible elements (Alpert (1993)). Various wavelet bases have been employed in the numerical treatment of integral equations. The ultimate aim is to see that the convergence is as rapid as possible. In addition to the conventional Daubechies wavelets, trigonometric wavelets, linear B-splines have been used [18], [19], [20]. These solutions are often complicated due to inherent difficulties involved in differentiation and integration of the underlying basis functions. Obviously, attempts to simplify procedures based on the wavelet approach are needed. In the present paper, rather than employing wavelet basis for $L_2(\mathbf{R})$ we use Haar wavelet packets that transform the dense matrices resulting from the discretization of 2nd kind integral equations into sparse matrices, a fact which enables the corresponding integral equations to be solved rapidly thereby, providing substantial improvement over classical and current methods. The main focus of the paper is to show how the

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use of Haar wavelet-packet series (HWPS) leads to sparsification of smooth integral operators on a finite interval, eventually leading to high order convergence. The paper is organized as follows. In section 2, a mathematical construction of new bases i.e. wavelet-packet bases is explained. In section 3, the basic properties of Haar wavelet packet series are discussed in some detail; followed by a brief explanation of how Haar wavelet-packet basis can be used to solve both Fredholm and Volterra linear integral equations. Section 4, describes the method of solution of nonlinear Volterra Hammerstein integral equation using single term Haar wavelet-packet series (STHWPS). Section 5, is devoted to convergence and error analysis. Numerical examples purely for illustrative purposes are discussed in Sec. 6.

2. Wavelet packets and wavelet packet Transform

A simple but powerful generalization of wavelets and the associated multiresolution analysis is wavelet packets. By generalizing the methods of multiresolution analysis, it is possible to construct orthonormal wavelet packets which provide a family of orthonormal basis of $L_2(\mathbf{R})$, which are related in some way to the classical Walsh functions. The set of Walsh functions is, in fact, the prototype of wavelet-packet basis of $L_2(\mathbf{R})$, just as the Haar wavelet system is the prototype of wavelet basis on $L_2(\mathbf{R})$. Wavelet packets being particular linear combinations of wavelets; they retain many of the properties such as orthonormality, smoothness and localization of their parent wavelets. An interesting observation is that, wavelet packets generated from Haar wavelets coincide with Walsh functions. To fix ideas and notations, we know that, given the bases functions $\{\phi_{1,k}(t)\}$ of V_1 from multiresolution analysis (MRA), $\{\phi(t-k)\}$ and $\{\psi(t-k)\}$ form orthonormal bases for V_0 and W_0 respectively and $V_1 = V_0 \oplus W_0$, here $\phi(t) = \sqrt{2} \sum_k h_k \phi(2t-k)$ and $\psi(t) = \sqrt{2} \sum_k g_k \psi(2t-k)$

This splitting trick can be used to decompose W spaces as well. For example, if we analogously define:

$$w_2(t) = \sqrt{2} \sum_k h_k \psi(2t-k)$$

and $w_3(t) = \sqrt{2} \sum_k g_k \psi(2t-k)$ then $\{w_2(t-k)\}, \{w_3(t-k)\}$

form orthonormal bases for the two subspaces whose direct sum is W_1 .

In general, for $n = 0, 1, \dots$, we define a sequence of functions as follows:

$$w_{2n}(t) = \sqrt{2} \sum_k h_k w_n(2t-k)$$

and

$$w_{2n+1}(t) = \sqrt{2} \sum_k g_k w_n(2t-k)$$

Clearly setting $n = 0$, we get $w_0(t) = \phi(t)$, the *father wavelet* and $n = 1$ yields $w_1(t) = \psi(t)$, the *mother wavelet*. Various combinations of these and their translations and dilations can give rise to variety of bases for the function spaces. So, we have a whole collection of orthonormal bases generated from $\{w_n(t)\}$. We call this collection "library of wavelet packets", and functions of the form $w_{nj k} = \sqrt{2} w_n \{2^j t - k\}$ are called *wavelet packets*.

As a particular case, we look at the wavelet packets generated from the Haar filters. Since the Haar filter has $h_0 = h_1 = \frac{1}{\sqrt{2}}$ and using

$$g_k = (-1)^k h_{1-k}, \quad g_0 = -g_1 = \frac{1}{\sqrt{2}}, \quad \text{we have}$$

$$w_{2n} = w_n(2t) + w_n(2t-1) \text{ and}$$

$w_{2n+1} = w_n(2t) - w_n(2t-1)$ with $w_0(t) = \chi_{[0,1]}$ the Haar scaling function and $w_1(t) = \chi_{[0,0.5]} - \chi_{[0.5,1]}$, the Haar wavelet. It indeed, turns out that $\{w_{nj k}\}$ are the well-known Walsh functions. Walsh proved that $\{w_{nj k}\}$ forms a complete orthonormal set (Fine, (1946). The full collection of Haar wavelet packets consists of translated and dilated Walsh functions and can be represented by $w_{nj k} = 2^{j/2} w_n(2^j t - k)$ where w_n is the n th Walsh function.

The wavelet packets transform generalizes the discrete wavelet transform and provides a more flexible tool for the time-scale analysis of functions. The wavelet transform is actually a subset of a far more versatile transformation, the wavelet packet transform. One step in the wavelet transform is that it calculates a low pass (scaling function) result and a high pass (wavelet function) result. The wavelet transform applies the wavelet transform step only to the low pass result. The wavelet packet transform applies the transform step to both the low pass and the high pass results. The *wavelet packet method* is a generalization of wavelet decomposition that offers a richer range of possibilities for signal analysis and synthesis. In wavelet analysis, a signal is split into an approximation and a detail. The approximation is then itself split into a second-level approximation and detail, and the process is repeated. For n -level decomposition, there are $n+1$ possible ways to decompose or encode the signal. We can visualize this from Fig.1.

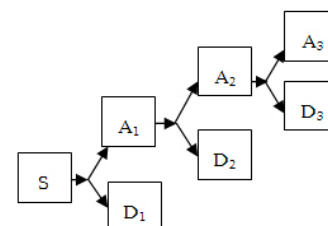


Fig .1: Wavelet decomposition

In wavelet packet analysis, the details as well as the approximations can be split. This yields more than $2^{2^{n-1}}$ different ways to encode the signal. This can be visualized from Fig 2. This is an example of a representation that is not possible with ordinary wavelet analysis (Andre Quinquis (1998)).

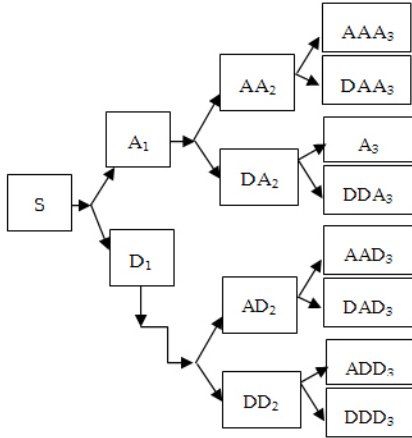


Fig. 2: Wavelet packet decomposition

3. Properties of Haar wavelet packet series (HWPS)

A function $f \in L^2([0,1])$, may be approximated using wavelet packets as

$$f(t) = \sum_i c_i w_i(t) \tag{3.1}$$

From the orthogonality property of wavelet packets, we have $c_i = \int_0^1 f(t)w_i(t)dt$ here $w_i(t)$ is i th

wavelet packet and c_i is the corresponding coefficient. In practice, only the first m terms are considered, where m is the integral power of 2. Then from (3.1), we get

$$f(t) = \sum_0^{m-1} c_i w_i(t) = \mathbf{c}_m^T \mathbf{w}_m(t),$$

here

$$\mathbf{c}_m = (c_0, c_1, \dots, c_{m-1})^T, \tag{3.2}$$

$$\mathbf{w}_m(t) = (w_0(t), w_1(t), \dots, w_{m-1}(t))^T. \tag{3.3}$$

The integration of a vector $\mathbf{w}_m(t)$ defined above can be approximated by

$$\int_0^t \mathbf{w}_m(\tau) d\tau = \mathbf{E} \mathbf{w}_m(t), \tag{3.4}$$

where $\mathbf{E}_{m \times m}$ is the operational matrix for integration (Chen and Hsiao (1975)) with $\mathbf{E}_{1 \times 1} = \frac{1}{2}$ and

$$\mathbf{E}_{m \times m} = \begin{bmatrix} \frac{1}{2} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & (-\frac{2}{m})^{j(m/8)} & \cdot & \cdot & \cdot \\ (\frac{2}{m})^{j(m/8)} & \cdot & 0(m/8) & \cdot & \cdot & \cdot \\ \cdot & (1/m)^{j(m/4)} & \cdot & 0(m/4) & \cdot & \cdot \\ \cdot & \cdot & (\frac{1}{2m})^{j(m/2)} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0(m/2) \end{bmatrix}$$

The Haar wavelet-packet transform matrix is defined as

$$H(1) = [1],$$

$$H(2) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$H(2^k) = \frac{1}{2^k} \begin{bmatrix} H(2^{k-1}) & H(2^{k-1}) \\ H(2^{k-1}) & -H(2^{k-1}) \end{bmatrix} = H(2) \otimes H(2^{k-1}), 2 \leq k \in \mathbf{N}$$

This is the matrix which corresponds to the *Fast Haar wavelet-packet transform*.

After developing the above computational tools, we shall proceed to explain the technique of solving linear integral equations. A linear Fredholm integral equation of second kind is of the form

$$f(x) = g(x) + \int_a^b K(x,t) f(t) dt \tag{3.5}$$

here K and g are given functions.

The kernel K is in $L^2[a,b]^2$ and f and $g \in L^2[a,b]$ We use the symbol K to denote the integral operator of (3.5), which is given by the formula

$$(Kf)(x) = \int_a^b K(x,t) f(t) dt, \tag{3.6}$$

$$\forall f \in L^2[a,b], x \in [a,b]$$

Now we pass on to the solutions based on the Haar wavelet-packets. Since the Haar wavelet-packets are defined only for the interval $[0,1]$, we must normalize equations (3.5) and (3.6). This can be done by the change of variable $t^* = (t-a)/(b-a)$

To solve (3.5), using HWPS approach; we represent f and g by their HWP series truncated to m terms:

$$f(x) = \sum_{i=0}^{m-1} f_i w_i(x) \text{ and } g(x) = \sum_{i=0}^{m-1} g_i w_i(x)$$

There are several methods by which we can approximate a kernel. In the proposed method, we expand the kernel $K(x,t)$ by a double HWP series (Shih and Han (1978)).

$$K(x,t) \approx \sum_0^{\infty} \sum_0^{\infty} k_{ij} w_i(x) w_j(t) \tag{3.7}$$

This series is convergent in the L_2 mean and the coefficients are given by

$$k_{ij} = \int_0^1 \int_0^1 K(x,t) w_i(x) w_j(t) dx dt.$$

An obvious choice of approximating kernel is the truncated expression

$$K(x, t) \approx \sum_0^{m-1} \sum_0^{n-1} k_{ij} w_i(x) w_j(t)$$

Because the series (3.7) is convergent, the coefficients k_{ij} are guaranteed to converge to zero as m, n increase. Now it is convenient to pass to a matrix-vector formulation of (3.1). Let K_{mm} be the matrix of the average values of $K(x, t)$ on all sub-squares of $[0,1] \times [0,1]$, a straightforward sampling of $K(x, t)$. This is a full matrix heavy to store, to multiply or to invert.

Next, our approach differs from the earlier work in that; it uses 2D-Haar wavelet packet transform (HWPT) and operational matrix of integration to approximate the integral operator:

We first approximate $K(x, t)$ using

$$K(x, t) \approx H_m K_{mm} H_m^T \quad (3.8)$$

This is the double wavelet packet series approximation of the kernel. Secondly,

$$(Kf)(x) \approx E_{m \times m}^T H_m K_{mm} H_m f_i E_{m \times m} \quad (3.9)$$

This is how a Haar wavelet-packet transform can be used to “sparsify” or “compress” operators in integral equations.

It is easy to see that, the original integral equation (3.5) can be approximated by a system of algebraic equations (Murlan Corrington (1973)),

$$(I - K) f = g. \quad (3.10)$$

Thus a Fredholm integral equation of 2nd kind is reduced to a finite matrix equation.

It may be noted that, a Volterra integral equation with convolutional kernel can be rewritten in Fredholm form.

4. Single-term Haar Wavelet packet series (STHWPS)

In this section we extend single term approach developed by Sepehrian and Razzaghi (2005), to Haar wavelet-packets for the numerical solution of nonlinear Volterra-Hammerstein integral equation of the form

$$f(x) = g(x) + \int_0^t k(t, s) h(s, f(s)) ds, \quad x \in [0,1], \quad (4.1)$$

$$\text{Let } z(s) = h(s, f(s)), \quad s \in [0,1]. \quad (4.2)$$

In order to solve (4.1) using STHWPS, we first divide the interval $[0,1]$ into m equal subintervals, where $m \in \mathbf{N}$. We then stretch each interval $i-1/m \leq t \leq i/m, i=1,2,\dots,m$, to $[0,1]$ by using transformations. $\tau_i = m t - (i-1)$ and $\lambda_i = m s - (i-1)$ we then have

$$f(\tau_i) = g(\tau_i) + \frac{1}{m} \int_0^{\tau_i} k(\tau_i, \lambda_i) z(\lambda_i) d\lambda_i; \quad (4.3)$$

and

$$f(\tau_i) = g(\tau_i) + \frac{1}{m} \sum_{j=1}^{i-1} \int_0^1 k(\tau_i, \lambda_j) z(\lambda_j) d\lambda_j + \frac{1}{m} \int_0^{\tau_i} k(\tau_i, \lambda_j) z(\lambda_j) d\lambda_j \quad (4.4)$$

Let $g(\tau_i)$ and $k(\tau_i, \lambda_j)$ be expressed by STHWPS

$$\text{as } g(\tau_i) = F^{(i)} w_0(\tau_i), \quad i=1,2,\dots,m. \quad (4.5)$$

$$k(\tau_i, \lambda_j) = K^{(i,j)} w_0(\tau_i) w_0(\lambda_j), \quad i=1,2,\dots,m, j=1,\dots,i, \quad (4.6)$$

$$\text{Here, } F^{(i)} = \int_{i-1/m}^{i/m} g(\tau_i) w_0(\tau_i) d\tau_i \text{ and}$$

$$K^{(i,j)} = m^2 \int_{i-1/m}^{i/m} \int_{m j-1/m}^{j/m} k(t, s) dt ds \quad (4.7)$$

Similarly, $f(s)$ and $z(s)$ are expanded by STHWPS as

$$f(\lambda_i) = Y^{(i)} w_0(\lambda_i), \quad (4.8)$$

$$\text{where } Y^{(i)} = \int_{i-1/m}^{i/m} f(\tau_i) w_0(\tau_i) d\tau_i$$

$$z(\lambda_i) = h(\lambda_i, Y^{(i)} w_0(\lambda_i)) \quad (4.9)$$

Let $z(\lambda_i)$ be expressed by STHWPS as

$$z(\lambda_i) = Z^{(i)} w_0(\lambda_i), \quad (4.10)$$

$$\text{Where } Z^{(i)} = \int_{i-1/m}^{i/m} z(\lambda_i) w_0(\lambda_i) d\lambda_i$$

Using $E = 1/2$ (Hsiao and Chen (1979)), and (4.6) and (4.10), we get

$$\int_0^{\tau_i} k(\tau_i, \lambda_j) z(\lambda_j) d\lambda_j = \frac{1}{2} K^{(i,i)} Z^{(i)} w_0(\tau_i), \quad i=1,\dots,m. \quad (4.11)$$

Using (4.3) and (4.11), the Block-pulse value $Y^{(1)}$ in the first interval is given by

$$Y^{(1)} = F^{(1)} + \frac{1}{2m} K^{(1,1)} Z^{(1)}, \quad (4.12)$$

And

similarly,

$$Y^{(i)} = F^{(i)} + \frac{1}{m} \sum_{j=1}^{i-1} K^{(i,j)} Z^{(j)} + \frac{1}{2m} K^{(i,i)} Z^{(i)}, \quad (4.13)$$

which is a system of nonlinear equations for $Y^{(i)}$,

Then by using

$$f(i) = 2Y^{(i)} - f(i-1), \quad i=1,2,\dots, \quad (4.14)$$

gives Block-pulse and discrete values of solution function $f(x)$.

5. Numerical Examples

We choose integral equations that can be solved analytically, so that accuracy and efficiency of the method can be checked easily. We consider Fredholm, Volterra transformed into Fredholm form, Fredholm-Hammerstein and Volterra-Hammerstein type of integral equations.

5.1 Consider, $f(x) = e^x - \frac{e^{x+1} - 1}{x+1} + \int_0^1 k(x,s)f(s)ds, x \in [0,1]$

a Fredholm type linear integral equation, with a symmetric kernel, $k(x,s) = e^{xs}$, $f(s) = e^s$.

The exact solution is $f(x) = e^x$.

This is solved by the method explained in section 4.3, for $m = 8$ and 16.

Now, $K_{8 \times 8}$ the matrix of the average values of $K(x,s)$ on all sub-squares of $[0,1] \times [0,1]$ is

| | | | | | | | |
|---------|---------|---------|---------|---------|---------|---------|---------|
| 1.00391 | 1.01179 | 1.01972 | 1.02772 | 1.03578 | 1.04391 | 1.05209 | 1.06034 |
| 1.01179 | 1.03578 | 1.06034 | 1.08549 | 1.11123 | 1.13758 | 1.16456 | 1.19218 |
| 1.01972 | 1.06034 | 1.10258 | 1.14651 | 1.19218 | 1.23967 | 1.28905 | 1.34041 |
| 1.02772 | 1.08549 | 1.14651 | 1.21095 | 1.27902 | 1.35091 | 1.42685 | 1.50705 |
| 1.03578 | 1.00023 | 1.19218 | 1.27902 | 1.37219 | 1.47214 | 1.57938 | 1.69443 |
| 1.04391 | 1.03758 | 1.23967 | 1.35091 | 1.47214 | 1.60425 | 1.74821 | 1.90509 |
| 1.05209 | 1.16456 | 1.28905 | 1.42685 | 1.57938 | 1.74821 | 1.35509 | 2.14195 |
| 1.06034 | 1.19218 | 1.34041 | 1.50705 | 1.69443 | 1.90509 | 2.14195 | 2.40826 |

From equation (3.8), $e^{xs} \approx H_{8 \times 8} K_{8 \times 8} H_{8 \times 8}$ this in expanded form is

| | | | | | | | |
|----------|---------|-----------|----------|----------|----------|---------|--------|
| 0.0129 | 0.0137 | 0.0147 | 0.0157 | 0.0168 | 0.0180 | 0.0184 | 0.0208 |
| -0.0002 | -0.0006 | -0.0011 | -0.0017 | 0.0023 | 0.0031 | 0.0030 | 0.0048 |
| 0 | 0.0001 | -0.00004 | -0.00009 | -0.0002 | -0.0003 | -0.0005 | 0.0006 |
| -0.0001 | -0.0003 | 0.00057 | 0.00085 | 0.00117 | -0.0015 | 0.0010 | 0.0024 |
| 0 | 0 | -0.00001 | 0.00002 | 0.00004 | -0.0008 | -0.0010 | 0.0001 |
| 0 | 0 | 0 | 0 | 0 | -0.00001 | -0.0009 | 0.0003 |
| 0 | 0 | 0.00002 | -0.00005 | 0.00008 | 0.00013 | -0.0011 | 0.0003 |
| -0.00005 | 0.0001 | -0.000287 | 0.00043 | -0.00058 | 0.00078 | 0.0019 | 0.0012 |

From equation (3.9),

$$\int_a^b K(x,t)f(t)dt \approx$$

| | | | | | | | |
|-----------|-----------|-----------|-----------|------------|----------|-----------|-----------|
| 0.006311 | -0.000067 | -0.000181 | -0.000113 | -0.0003999 | -0.00041 | -0.000440 | -0.000462 |
| -0.003202 | 0.000039 | 0.000122 | 0.000130 | 0.0002023 | 0.000212 | 0.000221 | 0.000232 |
| -0.001591 | 0.000017 | 0.000053 | 0.000054 | 0.000102 | 0.00010 | 0.000112 | 0.000121 |
| -0.000028 | 0 | -0.000033 | -0.000043 | 0 | 0 | 0 | -0.000151 |
| -0.000798 | 0 | 0.000291 | 0.000026 | 0.000050 | 0.000053 | 0.000057 | 0.000061 |
| 0.000016 | 0 | -0.000131 | -0.000023 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | -0.000011 | 0 | 0 | 0 | 0 |

We would like to draw attention to the above matrix. It is observed that, most of the entries are zero (or near to zero). Here the matrix contains only $O(n \log n)$ non-negligible elements proving the sparsity of the integral operator K (Alpert (1993)). This is one of the reasons for rapid convergence of the proposed method.

Also,

$$g(x) = e^x - \frac{e^{x+1} - 1}{x+1} \approx \left\{ \begin{matrix} g(1/16), g(3/16), g(5/16), g(7/16), \\ g(9/16), g(11/16), g(13/16), g(15/16) \end{matrix} \right\} \cdot H_{8 \times 8}$$

$$= \{-0.648092, -0.056142, 0.018611, -0.0296445, 0.004797, -0.0015815, 0.0093635, -0.015025\}.$$

The solution obtained by solving, Equation (3.10) is in wavelet-packet domain. By inverse transformation, the solution of the

original equation is obtained. Similar expressions can be obtained when $m = 16$.

Results obtained for $m=8$ and 16 are given in Table No. 1 and 2 respectively.

Table. 1. Comparison of HWPS solution with exact solution of Ex. 5.1, for $m = 8$

| x | y | HWPTS Solution | Error |
|-----|-------|----------------|-----------|
| 0 | 1.000 | 0.986 | 1.36e-002 |
| 0.1 | 1.105 | 1.091 | 1.40e-002 |
| 0.2 | 1.221 | 1.207 | 1.44e-002 |
| 0.3 | 1.350 | 1.335 | 1.47e-002 |
| 0.4 | 1.492 | 1.447 | 1.50e-002 |
| 0.5 | 1.649 | 1.663 | 1.53e-002 |
| 0.6 | 1.822 | 1.807 | 1.55e-002 |
| 0.7 | 2.014 | 1.998 | 1.56e-002 |
| 0.8 | 2.226 | 2.210 | 1.57e-002 |
| 0.9 | 2.246 | 2.244 | 1.57e-002 |
| 1 | 2.718 | 2.703 | 1.55e-002 |

Table. 2. Comparison of HWPS solution with exact solution of Ex.5.1, for $m=16$.

| x | y | HWPTS Solution | Error |
|-----|-------|----------------|-----------|
| 0 | 1.000 | 1.997 | 3.41e-003 |
| 0.1 | 1.105 | 1.102 | 3.50e-003 |
| 0.2 | 1.221 | 1.218 | 3.60e-003 |
| 0.3 | 1.350 | 1.346 | 3.68e-003 |
| 0.4 | 1.492 | 1.488 | 3.76e-003 |
| 0.5 | 1.649 | 1.645 | 3.82e-003 |
| 0.6 | 1.822 | 1.818 | 3.88e-003 |
| 0.7 | 2.014 | 2.010 | 3.91e-003 |
| 0.8 | 2.226 | 2.222 | 3.93e-003 |
| 0.9 | 2.460 | 2.456 | 3.92e-003 |
| 1 | 2.718 | 2.714 | 3.88e-003 |

5.2 Consider $f(x) = x + \int_0^x \sin(x-t)f(t)dt$ a Volterra equation with convolutional but nonsymmetrical kernel.

The exact solution is $y(x) = x + \frac{1}{6}x^3$

This can be rewritten in Fredholm form (Blyth

(2003)) as $f(x) = g(x) + \int_0^1 \bar{K}(x,t)f(t)dt$

here, $g(x) = x$ and $\bar{K}(x,t) = \begin{cases} \sin(x-t), & 0 \leq t < x \\ 0, & x \leq t < 1 \end{cases}$

This is solved by the method explained in section 3. Results obtained for $m=8$ and 16 are given in Table No. 3. The agreement between the exact solution

and the computed solution is impressive, even for taking small values for m . Numerical findings suggest that, the accuracy will improve dramatically by taking m large enough.

Table. 3. Comparison of HWPS solution with exact solution of Ex-5.2, for $m = 8$ and $=16$.

| x | y | Error($m=8$) | Error($m=16$) |
|-----|-----|----------------|-----------------|
| 0 | 1 | 5.12e-004 | 1.28e-004 |
| 0.1 | 1 | 5.65e-004 | 1.41e-004 |
| 0.2 | 1 | 6.25e-004 | 1.56e-004 |
| 0.3 | 1 | 6.90e-004 | 1.73e-004 |
| 0.4 | 1 | 7.63e-004 | 1.91e-004 |
| 0.5 | 1 | 9.32e-004 | 2.11e-004 |
| 0.6 | 1 | 1.03e-003 | 2.33e-004 |
| 0.7 | 1 | 1.14e-003 | 2.57e-004 |
| 0.8 | 1 | 1.26e-003 | 2.85e-004 |
| 0.9 | 1 | 1.32e-003 | 3.14e-004 |
| 1 | 1 | 1.39e-003 | 3.47e-004 |

5.3 Consider

$$f(x) - \int_0^1 k(x, s)g(s, f(s))ds = 1 - \frac{e^{x-1}}{2}(\sin 2 - \cos 2) - \frac{e^x}{2}(\cos 1 - \sin 1),$$

$x \in [0, 1]$, a Fredholm-Hammerstein nonlinear integral equation with convolutional kernel,

$k(x, s) = e^{x-s}$, $g(s, f(s)) = \cos(x + f(s))$ here the exact solution is $f(x) = 1$.

This is solved by the method explained in section 4. Results obtained with $m=10$ and 20 are given in Table No. 4.

Table. 4. Comparison of error estimates for $m=10$ and 20 of Ex-5.3.

| x | y | HWPS Solution ($m = 10$) | HWPS Solution ($m = 20$) |
|-----|---------|----------------------------|----------------------------|
| 0 | 0 | 0 | 0 |
| 0.1 | 0.10016 | 0.12334 | 0.10027 |
| 0.2 | 0.20133 | 0.21223 | 0.20145 |
| 0.3 | 0.30452 | 0.32043 | 0.30462 |
| 0.4 | 0.41066 | 0.42145 | 0.41075 |
| 0.5 | 0.52083 | 0.53156 | 0.52094 |
| 0.6 | 0.63600 | 0.64566 | 0.63623 |
| 0.7 | 0.71716 | 0.72654 | 0.71724 |
| 0.8 | 0.88533 | 0.89454 | 0.88548 |
| 0.9 | 1.02152 | 1.13452 | 1.02256 |
| 1 | 1.16667 | 1.22346 | 1.16785 |

5.4Next consider

$$f(t) = 1 + \sin^2(t) - 3 \int_0^t \sin(t-s) f^2(s) ds, \quad t \in [0, 1],$$

A Volterra-Hammerstein integral equation which is 'more' nonlinear than Example 5.3. This has the exact solution $f(t) = \cos t$ This equation is solved using the procedure explained in section 4.

The results obtained are plotted graphically. The graph shows that plots of STHWPS solution and exact solution are almost indistinguishable, even for small values of m .

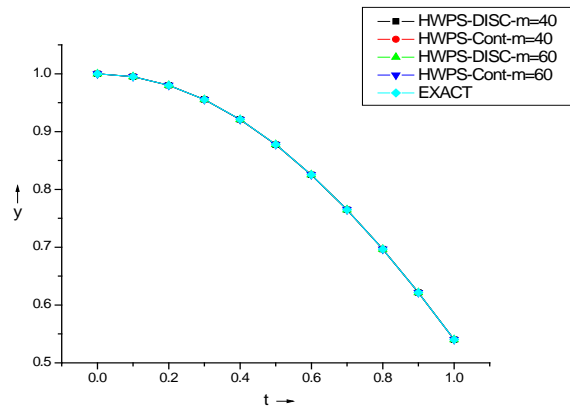


Fig. 3 Comparison of HWPS solution of example 5.4 with exact solution for $m = 40$ and $m = 60$.

6. Convergence and Error analysis

An important characteristic of any numerical method is not only its ability to guarantee convergence but equally important is the rate at which convergence is achieved. Approximation with Haar wavelet-packets is equivalent to the approximation with piecewise constant functions. If f, g and K in (3.5) are sufficiently smooth, then the convergence rate for piecewise constant functions is

$O(m^2)$. This property can be transferred to HWPS approach [20]. The efficiency of the method is demonstrated by some numerical examples; for getting error estimates, for which the exact solution $y_{ex}(x)$ is known are considered. The accuracy of the results is estimated by the error function $\epsilon_J = \max_{1 \leq i \leq m} (|y(x_i) - y_{ex}(x_i)|)$

So it is evidenced numerically that in the case of our solution, by halving the step size (doubling the resolution from $m=8$, to $m=16$) the error function roughly decreases quadratically. This theoretical estimation in general holds for the numerical finding in Tables 1-4. Results from Tables 1 and 2, show that the method yields rapid convergence with convergence rate ≈ 2 . Thus second order convergence is observed, as predicted by the theoretical considerations. The results of Table 4 show that STHWPS approach applied to example 3 is behaving as an order two scheme as expected from error analysis. All calculations are done using mathematica programs.

Conclusions

In this paper, the application of Haar wavelet-packet basis has been used for the solution of variety of second-kind integral equations in which integral operators are represented as sparse matrices which involves solution of the corresponding integral equations to be solved rapidly and accurately. These bases are very effective for the fast solution of wide class of such problems. The basic idea behind series and single-term approaches is to reduce the

problem to solving a system of linear or nonlinear algebraic equations. HWPS approach does not involve any integration since operational matrix of integration transforms integration into matrix-vector multiplication. By contrast, STHWPS approach does not require operational matrices of integration avoiding the possibility of computing and storing matrices of enormously large sizes. Furthermore, there is no restriction on m as in the case of series approach. Another major advantage of these new approaches is that besides meeting accuracy requirements, they lead to higher order convergence. These approaches have the additional advantage of $O(N)$ complexity, in the sense that the number of required operations is of the same order as the number of computed values. Our experience dictates that, many other classes of integral equations can be solved efficiently using these techniques.

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